More Perfect Numbers

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We are now equipped to prove a few new properties of perfect numbers.

- Let n be an odd perfect number.
  $\tau(n)$ is divisible by 2 (because n may not be a perfect square) but not 4.
  Suppose $\tau(n)$ were divisible by 4. Since $\tau(n) = \prod_i(\alpha_i + 1)$, that would indicate that at least two primes have odd multiplicities. Since all odd perfect numbers must be of the form $(4q+1)^{4c+1} \prod_i p_i^{2a_i}$, only one multiplicity will be odd ($4c+1$) and we have a contradiction. Therefore, $\tau(n)$ is not divisible by 4.

- No multiple greater than 1 of a perfect number, odd or even, is perfect.
  Let n be a perfect number.
  Since n is perfect, $\sigma(n) = 2n$. Let m be a natural number greater than 1. Due to the Fundamental Theorem of Arithmetic, m may be represented uniquely as the product of primes. We denote these primes $\{p_1, p_2, p_3, ...\}$.
  In order for nm to be perfect, $\sigma(nm) = 2nm$. We may rewrite m in terms of its prime factors to yield $\sigma(n * p_1 * p_2 * ...) = (2n * p_1 * p_2 * ...)$. Let us begin by multiplying n by p_1. According to the formula we proved earlier, $\sigma(n * p_1) = p_1\sigma(n) + \sigma(\frac{n}{p_1}) = 2n * p_1 + \sigma(\frac{n}{p_1})$. If $n * p_1$ were perfect, $\sigma(n * p_1) = 2n * p_1$. However, adding $\sigma(\frac{n}{p_1})$, which is always greater than 0, will cause np_1 to be abundant. Any multiple of an abundant number is also abundant, so we may stop here.