Ore Harmonic Numbers

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Oystein Ore defined the ore harmonic number as:

$$H(n) = \frac{\tau(n)}{\sum_{d|n} \frac{1}{d}}$$

However, we have presented it as:

$$H(n) = \frac{n * \tau(n)}{\sigma(n)}$$

We will now prove that the two statements are equivalent:

Proof: By definition, the variable d is a divisor of n. That means that there exists a natural number, which we denote d', for each d such that $dd' = n$. Note that d' is also a divisor of n, since it yields n when multiplied by an integer (d). In mathematical notation, $(\forall d\, \exists d')\, d' = \frac{n}{d}$.

Now let us multiply both sides of the fraction within the summation by d':

$$H(n) = \frac{\tau(n)}{\sum_{d|n} \frac{d'}{dd'}} = \frac{\tau(n)}{\sum_{d|n} \frac{d'}{n}}$$

We now have a common denominator of n, so we may move the summation into the numerator without changing the fraction:

$$H(n) = \frac{\tau(n)}{\left(\frac{\sum_{d|n} d'}{n}\right)}$$

Since every divisor d will have exactly one corresponding divisor d' and both are contained within the set of n's divisors, we have not altered the set at all; we have simply changed the ordering of the terms. As a result, $\sum_{d|n} d'$ is equivalent to $\sum_{d|n} d$ (and thus $\sigma(n)$), since addition is a commutative operation. Thus,

$$H(n) = \frac{\tau(n)}{\left(\frac{\sigma(n)}{n}\right)} = \frac{n * \tau(n)}{\sigma(n)}.$$