Perfect Numbers

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1 In General:

- Since perfect numbers must satisfy $\sigma(n) = 2n$ by definition, $\sigma(n)$ clearly must be even, and thus no perfect numbers are perfect squares or twice a square.
- All perfect numbers have integral Ore harmonic numbers (Ore).
- If m and n are perfect, $\sigma(m) * n = \sigma(n) * m$ $(2mn = 2nm)$.

2 Even Perfect Numbers:

Euclid conjectured, based on knowledge of the first four perfect numbers (6, 28, 496, and 8128), that all perfect numbers were of the form $(2^{p-1})(2^p - 1)$, where $(2^p - 1)$, and thus p, is prime. This conjecture appears in book 9 of the *Elements*. It was later proven by Euler for all *even* perfect numbers.

The number $(2^p - 1)$ is now known as a **Mersenne Prime**, after Marin Mersenne, a 17th century monk who listed many of these primes. Not all prime ps result in a Mersenne prime (for example, $2^{11} - 1 = 2047 = 23 * 89$), but all Mersenne primes will produce an even perfect number when used in Euclid's equation.

All even perfect numbers are both triangular and hexagonal. As a result, they can be written as the sum of the first $2^n - 1$ consecutive natural numbers.

3 Odd Perfect Numbers:

It is not known whether any odd perfect numbers exist, though it is highly improbable. If one does exist, it will meet certain conditions:

- If n is an odd perfect number, $2n$ will be 3-perfect ($\sigma(2n) = \sigma(2) * \sigma(n)$ $= 3\sigma(n) = 3 * 2n$). This is an extremely important property because it allows us to apply theorems and conjectures about 3-perfect numbers (such as the conjecture that there are finitely many 3-perfect numbers and they are all known) to odd perfect numbers as well.
- n is of the form $12j + 1$ or $36j + 9$ (Touchard).
- n is the sum of two squares (Stuyvaert).
- $\exists q, c \in \mathbb{N} \mid n = (4q + 1)^{4c+1} \prod_i p_i^{\alpha_i}$, where p_i is the ith distinct prime of n and α_i is its multiplicity (Euler).
- n is greater than 10^{300}. This lower bound will soon be increased to 10^{511} (Brent, Cohen, te Riele).