A Multiplication Theorem

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We will now present and prove a useful theorem concerning the effect of multiplication of the input of the divisor function on its result. We conjecture, based on the lack of assumptions about the value of the divisor function, that it is possible to find similar relations in all multiplicative functions.

Theorem: Whether or not p and n are relatively prime,

$$\sigma_k(pn) = p^k * \sigma_k(n) + \sigma_k\left(\frac{n}{p^\alpha}\right)$$

Where α is the multiplicity of the prime p in the set of n's divisors.

Proof: We can rewrite n as the product of some other natural number m and a power of the prime p equal to its multiplicity, α. That is, $(\exists m, p \in \mathbb{N})\, n = m * p^\alpha \wedge p \nmid m$. Dividing both sides by p^α, we see that $m = \frac{n}{p^\alpha}$. Since p is prime and does not divide m, m and p are relatively prime to each other.

That means that we can rewrite $\sigma_k(n)$ as $\sigma_k(p^\alpha m) = \sigma_k(p^\alpha) * \sigma_k(m)$.
By multiplying n by p, we add one to p's multiplicity:

$$\sigma_k(pn) = \sigma_k(p * p^\alpha m) = \sigma_k(p^{\alpha+1} m)$$

Since m and $p^{\alpha+1}$are relatively prime, $\sigma_k(p^{\alpha+1}m) = \sigma_k(p^{\alpha+1}) * \sigma_k(m)$.
Recall that $\sigma_k(p^\alpha) = \frac{p^{(\alpha+1)k}-1}{p^k-1}$ for all powers α of the prime p.
So $\sigma_k(pn) = \sigma_k(p^{\alpha+1}) * \sigma_k(m) = \frac{p^{[(\alpha+1)+1]k}-1}{p^k-1} * \sigma_k(m) = \frac{p^{(\alpha+2)k}-1}{p^k-1} * \sigma_k(m)$.
Similarly, $\sigma_k(n) = \sigma_k(p^\alpha) * \sigma_k(m) = \frac{p^{(\alpha+1)k}-1}{p^k-1} * \sigma_k(m)$.
Let us rewrite $\frac{p^{(\alpha+2)k}-1}{p^k-1}$ as $\frac{p^{(\alpha+2)k}-p^k+(p^k-1)}{p^k-1}$.
Clearly, these are equivalent, since $-p^k$ and p^k will cancel each other out.
Now we may factor out a p^k, leaving us with $\sigma_k(pn) = \frac{p^k(p^{(\alpha+1)k}-1)+(p^k-1)}{p^k-1} * \sigma_k(m) = \left(\frac{p^k(p^{(\alpha+1)k}-1)}{p^k-1} + 1\right) * \sigma_k(m) = \frac{p^k(p^{(\alpha+1)k}-1)}{p^k-1} * \sigma_k(m) + \sigma_k(m)$. Since $\sigma_k(n) = \frac{p^{(\alpha+1)k}-1}{p^k-1} * \sigma_k(m)$, we can substitute this into $\sigma_k(pn)$:

$$\sigma_k(pn) = p^k * \sigma_k(n) + \sigma_k(m)$$

Since $m = \frac{n}{p^\alpha}$, as stated above, $\sigma_k(pn) = p^k * \sigma_k(n) + \sigma_k\left(\frac{n}{p^\alpha}\right)$ ∎